10.4 Polar Coordinates and Polar Graphs

**Pole (or origin):** a fixed point, O, in a plane around which the polar coordinate system is constructed.

**Polar Axis:** a horizontal ray initiating at O and going to the right.

Any point P in the plane can be assigned a polar coordinate \((r, \theta)\).

- \(r\): the directed distance from \(O\) to \(P\).
- \(\theta\): the directed angle, counterclockwise from the polar axis to the segment \(OP\).

**Radial lines:** lines through the origin indicating the radius through concentric circles.

Unlike rectangular coordinates, in polar coordinates the points are not unique. That is, \((r, \theta)\) and \((r, \theta+2\pi)\) represent the same point.

In fact, because \(r\) is a directed distance, \((-r, \theta)\) represents the same point as \((r, \theta+\pi)\).

As a rule, \((r, \theta) = (r, \theta+2n\pi)\).

Also, \((r, \theta) = (-r, \theta+(2n+1)\pi)\).

**Theorem 10.10 Coordinate Conversion**
The polar coordinates \((r, \theta)\) of a point are related to the rectangular coordinates \((x, y)\) of the point as follows:

1. \(x = r \cos \theta\)
2. \(\tan \theta = \frac{y}{x}\)
3. \(y = r \sin \theta\)
4. \(r^2 = x^2 + y^2\)
Ex 1 Plot the points in polar coordinates and find the corresponding rectangular coordinates for the point.

a) \((2, \frac{7\pi}{4})\)  
b) \((-4, -\frac{7\pi}{6})\)

Ex 2 Use the angle feature of a graphing utility to find the rectangular coordinates for the point given in polar coordinates.

\((-2, \frac{11\pi}{6})\)

Ex 3 The rectangular coordinates of a point are given. Plot the point and find two sets of polar coordinates for the point for \(0 \leq \theta < 2\pi\).

\((4, -2)\)
Ex 4 Use the angle feature of a graphing utility to find one set of polar coordinates for the point given in rectangular form. 
(3, -2)

Ex 5 Convert the rectangular equation to polar form and sketch its graph.
\[ x^2 - y^2 = 9 \]

Ex 6 Convert the polar equation to rectangular form and sketch its graph.
\[ r = 5 \cos \theta \]

Ex 7 Use a graphing utility to graph the polar equation. Find an interval for \( \theta \) over which the graph is traced \textit{only once}.
\[ r = 3(1 - 4 \cos \theta) \]
**Side Note:**

**Spiral of Archimedes:** the graph of the parametric equations

\[
x = \frac{1}{2} \theta \cos \theta \\
y = \frac{1}{2} \theta \sin \theta
\]

This can be represented in polar form by \( r = a \theta \).

**Theorem 10.11 Slope in Polar Form**

If \( f \) is a differentiable function of \( \theta \), then the slope of the tangent line to the graph of \( r = f(\theta) \) at the point \((r, \theta)\) is

\[
\frac{dy}{d\theta} = \frac{f(\theta) \cos \theta + f'(\theta) \sin \theta}{-f(\theta) \sin \theta + f'(\theta) \cos \theta}
\]

provided that \( \frac{dx}{d\theta} \neq 0 \) at \((r, \theta)\).

**Note:**

1. Solutions to \( \frac{dy}{d\theta} = 0 \) yield horizontal tangents, provided that \( \frac{dx}{d\theta} \neq 0 \).
2. Solutions to \( \frac{dx}{d\theta} = 0 \) yield vertical tangents, provided that \( \frac{dy}{d\theta} \neq 0 \).

If \( \frac{dy}{d\theta} \) and \( \frac{dx}{d\theta} \) are simultaneously 0, no conclusions can be drawn about tangent lines.

**Theorem 10.12 Tangent Lines at the Pole**

If \( f(\alpha) = 0 \) and \( f'(\alpha) \neq 0 \), then the line \( \theta = \alpha \) is tangent at the pole to the graph of \( r = f(\theta) \).

Ex 8 Sketch a graph of the polar equation and find the tangents at the pole.

\[ r = 3 \cos 2\theta \]
Example 9 Sketch a graph of the polar equation.

a. \( r = 1 \)

\[ \text{Graph of } r = 1 \]

b. \( r = 1 + \sin \theta \)

\[ \text{Graph of } r = 1 + \sin \theta \]

See Special Polar Graphs on Page 737.

10.4 Homework # 1, 5, 9, 11, 15, 24, 27, 31, 35, 59, 47, 77, 81, 85, 87